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Variations on ward continuity in 2-normed spaces

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Abstract. In this paper, the concept of a quasi-Cauchy sequence is generalized to a concept of a p-quasi-Cauchy sequence for any fixed positive integer p in a 2-normed space X. Some interesting theorems related to p-ward continuity and uniform continuity are obtained. A sequence (x_n) in a 2-normed space X is called p-quasi-Cauchy if $lim_{n\to\infty}\|x_{n+p} - x_n, z\| = 0$ for all $z \in X$. It turns out that if a function f defined on a subset of X is uniformly continuous, then f preserves p-quasi-Cauchy sequences for all positive integer p.
Keywords: sequences, series, summability, continuity, compactness, 2-normed space
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Introduction

S. Gähler introduced the concept of 2-normed spaces in 1960 ([1], [2]) and then a lot of interesting improvements have been done in 2-normed spaces by many different authors, see for instance ([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). Let X be a real vector space of dimension d, where $dimX > 1$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}^+$ which satisfies: (i) $\|x, y\| = 0 \Leftrightarrow x$ and y are linearly dependent, (ii) $\|x, y\| = \|y, x\|$, (iii) $\|\alpha x, y\| = |\alpha|\,\|x, y\|$ for all $\alpha \in \mathbb{R}$, (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. Then $(X, \|.,.\|)$ is called a 2-normed space.

Recently, the concept of quasi-Cauchy sequences is introduced by Burton and Coleman in [14] and then using the idea of quasi Cauchy sequences, Cakalli has defined the concepts of ward continuity and ward compactness for functions in [15]. A real function is called ward continuous if $\lim_{n\to\infty} \Delta f(x_n) = 0$ whenever $\lim_{n\to\infty} \Delta x_n = 0$, i.e. f preserves quasi-Cauchy sequences; and a subset E of $\mathbb{R}$ is called ward compact if any sequence (x_n) of points in E has a quasi-Cauchy subsequence. Since then different types of continuities have been introduced by Cakalli in [16, 17, 18, 19, 20].

In [21], the concept of ward continuity and the concept of ward compactness have been introduced in 2-normed spaces. A function f is called ward continuous if it preserves quasi-Cauchy sequences, i.e. $lim_{n\to\infty}\|\Delta f(x_n), w\| = 0$ for every $w \in X$ whenever $lim_{n\to\infty}\|\Delta x_n, z\| = 0$ for every $z \in X$ and a subset E of 2-normed space X is called ward compact if any sequence of points in E has a quasi-Cauchy subsequence. Also some other kinds of continuity in 2-normed space are investigated in [22, 23, 24].

The concept of p-quasi-Cauchy sequences of points in $\mathbb{R}$ is defined by Cakalli in [25]. The purpose of this paper is to generalize the concept of ward continuity to the concept of p-ward continuity replacing $\Delta x_n = x_{n+1} - x_n$ with $\Delta x_n = x_{n+p} - x_n$ for any fixed positive integer p in a 2-normed space and investigate some results about p-quasi-Cauchy continuity and uniform continuity.

p-ward continuity in 2-normed space

Now we give the definition of p-quasi-Cauchy sequences of points in a 2-normed space.

Definition 1 *Let p be a fixed positive integer. A sequence (x_n) is called p-quasi-Cauchy in X if* $\lim_{n\to\infty} \|\Delta_p x_n, z\| = 0$*, where $\Delta_p x_n = x_{n+p} - x_k$ and for every $z \in X$.*

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For a special case if $p = 1$, then p-quasi-Cauchy sequences turns to be quasi-Cauchy sequences, i.e. 1-quasi-Cauchy sequences are quasi-Cauchy sequences. We will denote the set of all p-quasi-Cauchy sequences by Δ_p. From the equality

$$x_{n+p} - x_n = x_{n+p} - x_{n+p-1} + x_{n+p-1} - x_{n+p-2} ... - x_{n+2} + x_{n+2} - x_{n+1} + x_{n+1} - x_n,$$

it is obvious that any quasi-Cauchy sequence is also p-quasi-Cauchy, but the converse is not true; the counterexample is given by the sequence $((-1)^n x)$.

The sum of two p-quasi-Cauchy sequence is p-quasi-Cauchy. Let (x_n) and (y_n) be p-quasi-Cauchy sequences in X. So $\lim_{n\to\infty} \|\Delta_p x_n, z\| = 0$ and $\lim_{n\to\infty} \|\Delta_p y_n, z\| = 0$ for every $z \in X$. Therefore $\lim_{n\to\infty} \|\Delta_p(x_n + y_n), z\| \leq \lim_{n\to\infty} \|\Delta_p x_n, z\| + \lim_{n\to\infty} \|\Delta_p y_n, z\| = 0$. Moreover for any constant $c \in X$ then (cx_n) is p-quasi-Cauchy sequence in X. So the set E of p-quasi-Cauchy sequences in X is a vector space.

Here is the definition of the p-ward compactness of a subset of X.

Definition 2 *Let E be the subset of X and (x_n) be a sequence of points in E. The set E is called p-ward compact if any sequence of points in E has a p-quasi-Cauchy subsequence of (x_n).*

Let E be a ward compact set and (x_n) be the sequence of points in $A \subseteq E$. Since $A \subseteq E$, the terms of the sequence (x_n) are contained in the set of E. Since E is a ward compact set, there is a subsequence (x_{n_k}) of (x_n). So the set A is ward compact. Since any quasi-Cauchy sequence is also p-quasi-Cauchy, then we see that any ward compact subset of X is p-ward compact.

Similarly it can be easily shown that the intersection of any p-ward compact subsets of X is p-ward compact. Furthermore any subset of a p-ward compact set of X is p-ward compact and any bounded subset of X is p-ward compact.

Now we define the new type continuity as p-ward continuity or p-quasi-Cauchy continuity:

Definition 3 *A function f is called p-ward continuous or p-quasi-Cauchy continuous on the subset E of X if it preserves p-quasi-Cauchy sequences, i.e. $lim_{n\to\infty}\|\Delta_p f(x_n), w\| = 0$ for every $w \in X$ whenever $lim_{n\to\infty}\|\Delta_p x_n, z\| = 0$ for every $z \in X$.*

Theorem 1 *If f is p-ward continuous on a subset E of X, then it is continuous on E.*

We note that the set of p-ward continuous functions is a vector space. It is easy to evaluate the sum of two p-quasi-Cauchy continuous functions is p-quasi-Cauchy continuous in a 2-normed space and for any constant c, cf is p-ward continuous whenever f is.

Theorem 2 *If f is p-ward continuous on a subset E of X, then it is ward continuous on E.*

Theorem 3 *p-ward continuous image of any p-ward compact subset of X is p-ward compact.*

We also note that p-ward continuous image of any compact subset of X is compact. The result can be easily obtained from the previous theorem. The uniform continuity in 2-normed space is defined as: For any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|f(x) - f(y), w\| < \epsilon$ for any $w \in X$ whenever $\|x - y, z\| < \delta$ for any $x, y, z \in X$. The next theorem gives the relation between p-ward continuity and uniform continuity on X. The theorem states that any uniformly continuous function preserves p-quasi-Cauchy sequences on X.

Theorem 4 *If f is uniformly continuous on a subset E of X, then it is p-ward continuous on E.*

Theorem 5 *If (f_n) is a sequence of p-ward continuous functions defined on a subset E of X and (f_n) is uniformly convergent to a function f, then f is p-ward continuous on E.*

Theorem 6 *The set of all p-ward continuous functions defined on a subset E of X is a closed subset of the set of all continuous functions on E.*

Conclusion

In this paper, the concept of p-ward continuity in 2-normed space X and also a concept of p-ward compactness of a subset E of X are introduced. Some relations between continuity, ward continuity, p-ward continuity and uniform continuity on X are investigated. In [21], there are some results that we obtain for the special case $p = 1$. For a further study, we suggest to investigate p-quasi-Cauchy double sequences of points in X.

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