International Journal of Geometric Methods in Modern Physics
(2026) 2650263 (14 pages)
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DOI: 10.1142/S0219887826502634

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Exact vacuum FLRW solutions in q-deformed Brans–Dicke cosmology

Salih Kibaroğlu *

Maltepe University, Faculty of Engineering and Natural Sciences, 34857, Istanbul, Türkiye
salihkibaroglu@maltepe.edu.tr

Mustafa Senay

Department of Medical Services and Techniques
Vocational School of Health Services
Bartın University, 74100, Bartın, Türkiye
msenay@bartin.edu.tr

Received 12 March 2026
Accepted 27 May 2026
Published 23 June 2026

We study a q-deformed extension of Brans–Dicke (BD) gravity in a spatially flat Friedmann–Lemaître–Robertson–Walker space-time. The deformation enters through a coupling function that modifies the effective gravitational strength and leads to generalized Friedmann equations. In the matter-free sector, we obtain exact analytic solutions for the scale factor and the BD scalar field, and recast the scalar contribution as an effective fluid. We show that the corresponding equation-of-state parameter and the deceleration parameter are constants and depend only on the BD coupling ω and the deformation function, allowing the scalar sector to mimic radiation-, matter-, or dark-energy-like behavior for a restricted region of parameter space.

Keywords: Brans–Dicke gravity; cosmological solutions; q-deformation; thermodynamic.

1. Introduction

The gravitational interaction may receive corrections beyond General Relativity when new degrees of freedom, thermodynamic arguments, or quantum/statistical effects are taken seriously. A broad class of such modifications is provided by scalar–tensor theories [1–4], where the effective gravitational coupling becomes dynamical. Brans–Dicke (BD) theory [5] is the prototypical example, and has been applied to cosmology and compact objects in many contexts, including black holes [6–8], interactions with matter fields [9], time-varying gravitational coupling and dark

*Corresponding author.

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sector phenomenology [10–13], scale-invariant inflationary scenarios [14–16], non-Riemannian formulations [8, 17], gravity waves [17], and parameter-space studies of ω and observational constraints [18].

In parallel, the thermodynamic interpretation of gravitation triggered by black hole entropy and Hawking radiation [19, 20] and later developed in various forms [21–23] has motivated derivations of cosmological equations from horizon thermodynamics [24–29] and, more broadly, emergent/entropic gravity ideas [30, 31]. Within this viewpoint, it is natural to ask how nonstandard microscopic statistics feeds into macroscopic gravitational dynamics.

Deformed-statistics frameworks (including q-deformations) provide a systematic way to encode nonstandard microscopic behavior and have been used to obtain modified Einstein/Friedmann equations and related cosmological consequences [32–38]. However, the interplay between (i) a genuinely scalar–tensor gravitational sector in the BD sense and (ii) q-deformation emerging from an entropic-gravity construction has not been fully explored at the level of explicit cosmological solutions and their effective fluid interpretation. In particular, it is not obvious *a priori* whether the resulting scalar sector can mimic standard cosmological phases (radiation/matter/dark energy), or whether it introduces qualitatively new expansion histories.

In this work, we study a q-deformed extension of BD gravity — obtained previously from Verlinde-type entropic arguments [39] — in a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) background. The deformation enters through a coupling function $\alpha(z, q)$ and modifies the effective gravitational coupling. The paper is organized as follows. In Sec. 2, we review the q-deformed BD field equations obtained from the entropic-gravity setup. In Sec. 3, we derive the corresponding FLRW equations and study their exact solutions in the matter-free sector, together with the effective EoS and deceleration parameter. We conclude in Sec. 4.

Notations and conventions: In this work, we adopt the mostly negative metric signature, such that the four-dimensional Minkowski metric takes the form $\mathrm{diag}(1, -1, -1, -1)$. Throughout the paper, we work in natural units by setting the fundamental physical constants to unity: the speed of light $c = 1$, and Boltzmann's constant $k_B = 1$. The four-dimensional space-time indices are represented by Roman letters such as $a, b, c, ... = 0, 1, 2, 3$.

2. q-Deformed Brans–Dicke Theory

Here, we briefly summarize the formulation introduced in [39], where BD gravity is extended by q-deformed statistics within Verlinde's entropic-gravity framework.

2.1. *q-deformed background*

The deformation is incorporated in the entropy-area relation and therefore in the local Unruh temperature on the holographic screen. Using the conformal transformation $\tilde{g}_{ab} = \psi g_{ab}$, with ψ the BD scalar, the local temperature measured by an

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accelerated observer becomes [40–42]

$$T = \alpha\left(z,q\right)\frac{\hbar}{2\pi}e^{\phi}N^{a}\left(a_{a}+\frac{1}{2}\nabla_{a}\log\psi\right), \tag{1}$$

where $\phi = \frac{1}{2}\log\sqrt{-\xi^{a}\xi_{a}}$ is the relativistic gravitational potential (ξ^{a} is a time-like Killing field), accordingly, the exponential factor e^{ϕ} represents the redshift between local and asymptotic observers, with $\phi = 0$ taken as the reference point at spatial infinity, $a^{b} = -\nabla^{b}\phi$ is the Jordan-frame acceleration, and N^{a} is the outward unit normal. The function $\alpha\left(z,q\right)$ encodes the statistical deformation:

$$\alpha\left(z,q\right) = \frac{5N}{2}\left[\frac{5}{2}\frac{f_{5/2}(z,q)}{f_{3/2}(z,q)} - \ln z\right], \tag{2}$$

where $0 < q < 1$ measures the deformation strength, $z = \exp\left(\mu/T\right)$ is the fugacity (μ is the chemical potential) and N is the total particle number,

$$N = \sum_{i}\frac{1}{|\ln q|}\left|\ln\left(\frac{|1-ze^{-x_{i}}|}{1+qze^{-x_{i}}}\right)\right|, \tag{3}$$

where $x_{i} = \beta\epsilon_{i}$ is a dimensionless variable representing the energy scale of the system, introduced to simplify the evaluation of thermostatistical functions and integrals. Here, $\beta = 1/k_{B}T$ denotes the inverse thermal energy, and ϵ_{i} represents the kinetic energy of a particle in the ith energy state. The function $f_{n}(z,q)$ denotes the generalized Fermi–Dirac function,

$$\begin{aligned} f_{n}(z,q) &= \frac{1}{\Gamma(n)}\int_{0}^{\infty}\frac{x^{n-1}}{|\ln q|}\left|\ln\left(\frac{|1-ze^{-x}|}{1+qze^{-x}}\right)\right|\mathrm{d}x \\ &= \frac{1}{|\ln q|}\left[\sum_{l=1}^{\infty}(-1)^{l-1}\frac{(zq)^{l}}{l^{n+1}} - \sum_{l=1}^{\infty}\frac{z^{l}}{l^{n+1}}\right], \end{aligned} \tag{4}$$

where $\Gamma\left(n\right) = \int_{0}^{\infty}x^{n-1}e^{-x}dx$ is the Gamma function for $n > 0$ and $x = \beta\epsilon$.

For Eqs. (3) and (4), different regimes of x characterize the relative contribution of energy states. In the limit $x \ll 1$, low-energy states dominate and the deformation effects associated with q become more pronounced. In contrast, for $x \gg 1$ the factor e^{-x} strongly suppresses high-energy contributions. The intermediate regime $x \approx 1$ corresponds to the characteristic thermal scale where particle energies are comparable to the thermal energy and therefore contribute most significantly to the generalized Fermi–Dirac integrals appearing in $\alpha\left(z,q\right)$. In this work, we focus on this regime in order to capture the dominant thermodynamic behavior of the system rather than the extreme limits.

In Fig. 1, we illustrate the behavioral analysis of the statistical deformation function $\alpha\left(z,q\right)$ with respect to various values of the deformation parameter q and fugacity z in the regime $x \approx 1$. Furthermore, $\alpha\left(z,q\right)$ spans a wide range, from values close to zero to very large magnitudes ($\sim 10^{7}$) as both q and z approach unity. This wide variability allows $\alpha\left(z,q\right)$ to be interpreted as an effective cosmological free parameter in our model.

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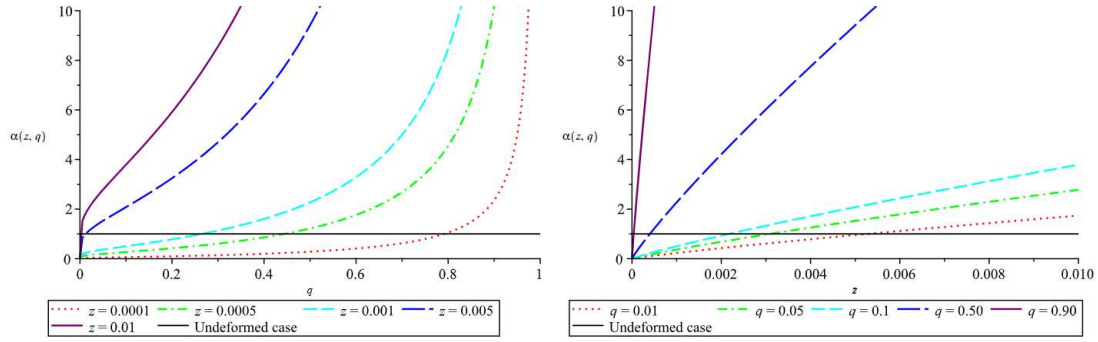

(a) (b)

Fig. 1. Panel (a) shows $\alpha(z,q)$ as a function of the deformation parameter q for fixed values of the fugacity z. Panel (b) shows $\alpha(z,q)$ as a function of the fugacity z for fixed values of the deformation parameter q. The undeformed case $\alpha(z,q)=1$ is shown as the black horizontal reference line.

2.2. *Generalized gravitational field equations*

By invoking the equipartition theorem on the holographic screen, we derived the total mass M which can be seen as a modified version of Gauss's law under scalar–tensor gravity

$$M=\frac{\alpha\left(z,q\right)}{4\pi}\oint_{\partial\Sigma}e^{\phi}N^{a}\left(a_{a}+\frac{1}{2}\nabla_{a}\log\psi\right)\psi\mathrm{d}A, \tag{5}$$

where $\mathrm{d}A$ is the surface area element on $\partial\Sigma$. Using Stokes' theorem and matching with the Komar definition gives the generalized field equations

$$R_{ab}-\frac{1}{2}g_{ab}R=\frac{8\pi}{\alpha\left(z,q\right)\psi}\left(T_{ab}^{M}+T_{ab}^{\psi}\right), \tag{6}$$

with matter contribution T_{ab}^{M} and scalar contribution

$$T_{ab}^{\psi}=\frac{1}{8\pi}\left[\frac{\omega}{\psi}\left(\partial_{a}\psi\partial_{b}\psi-\frac{1}{2}g_{ab}\partial_{c}\psi\partial^{c}\psi\right)+\alpha\left(z,q\right)\left(\nabla_{a}\nabla_{b}-g_{ab}\Box\right)\psi\right], \tag{7}$$

where $\Box\equiv g^{ab}\nabla_{a}\nabla_{b}$ denotes the d'Alembert operator and ω is a dimensionless coupling parameter analogous to the BD parameter. Additionally, Eq. (6) can be written as $G_{ab}=8\pi G_{\mathrm{eff}}\left(T_{ab}^{M}+T_{ab}^{\psi}\right)$ with

$$G_{\mathrm{eff}}=\left[\alpha\left(z,q\right)\psi\right]^{-1}, \tag{8}$$

which makes explicit how the deformation and scalar sector jointly renormalize the gravitational strength. However, in our equations, we prefer to use the exact form to see the effect of the deformation parameter explicitly.

Consistency limits are immediate: (i) for $\alpha\left(z,q\right)=1$ one recovers standard BD theory [5]; (ii) for $\psi=1$ one obtains the q-deformed Einstein equations [32–34, 36]; and (iii) imposing both $\alpha=1$ and $\psi=1$ yields Einstein gravity.

In Sec. 3, we specialize these equations to a spatially flat FLRW background.

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3. Cosmological Analysis

In general, for most cosmological models, the metric is assumed to be a flat FLRW metric which is based on Einstein's equations of general relativity. This framework facilitates the description of the universe's evolution under the assumptions of homogeneity and isotropy on large scales. Accordingly, we begin with the following line element:

$$ds^2 = dt^2 - a\left(t\right)^2\left(dx^2 + dy^2 + dz^2\right), \tag{9}$$

where $a\left(t\right)$ is the scale factor. Consistent with our assumptions, we consider the scalar field as a function of cosmic time, denoted as $\psi\left(x\right) \to \psi\left(t\right)$. Accordingly, the energy–momentum tensors T^{M}_{ab} and T^{ψ}_{ab} can be consistently interpreted as those of classical perfect fluids, as they take the standard diagonal form in a comoving frame:

$$T^{M\,a}_{\ \ b} = \mathrm{diag}\left(\rho_M, -P_M, -P_M, -P_M\right), \tag{10}$$

and

$$T^{\psi\,a}_{\ \ b} = \mathrm{diag}\left(\rho_\psi, -P_\psi, -P_\psi, -P_\psi\right), \tag{11}$$

where ρ_M, ρ_ψ denote the energy densities and P_M, P_ψ represent the corresponding isotropic pressures of the matter and scalar field components, respectively. These tensors are consistent with the general form of a perfect fluid energy–momentum tensor, $T_{ab} = \left(\rho + P\right)u_a u_b - P g_{ab}$ where u_a is the fluid's four-velocity. Furthermore, each fluid component can be characterized by an EoS parameter defined as $W_i = P_i/\rho_i$, with $i = M, \psi$, which encapsulates the thermodynamic relation between pressure and energy density. This framework allows for a straightforward cosmological interpretation of both matter and scalar field contributions in terms of perfect fluids.

Having established the necessary background, we can now proceed to derive the Friedmann equations. The following equation can be obtained by focusing on the $(0,0)$ component of Eq. (6)

$$3H^2 = \frac{8\pi}{\alpha\left(z,q\right)\psi}\rho_M + \frac{\omega}{2}\frac{\dot{\psi}^2}{\alpha\left(z,q\right)\psi^2} - 3H\frac{\dot{\psi}}{\psi}, \tag{12}$$

where $H\left(t\right) = \dot{a}\left(t\right)/a\left(t\right)$ is the Hubble parameter. The (i,i) components of Eq. (6) lead to,

$$2\dot{H} + 3H^2 = -\frac{8\pi}{\alpha\left(z,q\right)\psi}P_M - \frac{\omega}{2\alpha\left(z,q\right)}\frac{\dot{\psi}^2}{\psi^2} - 2H\frac{\dot{\psi}}{\psi} - \frac{\ddot{\psi}}{\psi}, \tag{13}$$

where the dot denotes the derivative with respect to the cosmic time (d/dt) and the pressure P_ψ and energy density ρ_ψ parameters can be written as

$$\rho_\psi = \frac{1}{8\pi}\left[\frac{\omega}{2}\frac{\dot{\psi}^2}{\psi} - 3\alpha\left(z,q\right)H\dot{\psi}\right], \tag{14}$$

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and

$$P_\psi = \frac{1}{8\pi}\left[\frac{\omega}{2}\frac{\dot{\psi}^2}{\psi} + \alpha(z,q)\left(2H\dot{\psi} + \ddot{\psi}\right)\right]. \tag{15}$$

Then, by making use of Eqs. (12) and (13), the acceleration equation can be obtained as

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3\alpha(z,q)\,\psi}\left[3\left(P_M + P_\psi\right) + \rho_M + \rho_\psi\right]. \tag{16}$$

These are the deformed Friedmann equations in the context of q-deformed BD gravitation theory. For completeness, if we combine Eqs. (12) and (16), we also derive the time derivative of the Hubble parameter as

$$\dot{H} = -\frac{4\pi}{\alpha(z,q)\,\psi}\left(P_M + \rho_M\right) - \frac{\omega}{2\alpha(z,q)}\left(\frac{\dot{\psi}}{\psi}\right)^2 + \frac{H}{2}\frac{\dot{\psi}}{\psi} - \frac{\ddot{\psi}}{2\psi}. \tag{17}$$

This equation provides a detailed description of the evolution of the Hubble parameter, including contributions from matter, the scalar field, and the deformation function $\alpha(z,q)$.

3.1. *Solutions for the Friedmann equations*

We now turn to the analysis of the Friedmann equations in the absence of ordinary matter. In this vacuum configuration, the cosmic dynamics are governed entirely by the modified gravitational sector, namely the scalar field and the q-deformation effects, without contributions from standard matter or radiation components. This setting enables us to isolate and clearly identify the pure impact of the q-deformed scalar–tensor structure on the cosmological evolution. Under these assumptions, solving Eq. (12) for the scalar field $\psi(t)$ yields the following relation:

$$\psi(t)^{\pm} = \psi_0 a(t)^{\frac{\mathcal{A}^{\pm}+3\alpha}{\omega}}, \tag{18}$$

where ψ_0 is a constant and

$$\mathcal{A}^{\pm} = \pm\sqrt{3\alpha(2\omega+3\alpha)}, \tag{19}$$

is introduced for simplicity (here α is the shorter representation of the deformation function $\alpha(z,q)$). Here, it is important to note that in order to avoid complex values, the parameter ω has to satisfy the following condition:

$$\omega \geq -\frac{3}{2}\alpha. \tag{20}$$

In analogy with BD theory, we have demonstrated that the scalar field evolves as a power-law function of the scale factor $a(t)$, consistent with theoretical expectations (see [43–46]). Notably, in [47], this power-law behavior of the BD scalar was assumed to explore the dynamics of BD cosmology with a cosmological term.

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Furthermore, it has been observed that adopting this power-law form for the BD scalar significantly enhances the fit to cosmological data [48].

Moreover, substituting Eq. (18) into Eq. (13) and solving with respect to $a(t)$ we get

$$a(t)^{\pm} = \left[\mathcal{B}^{\pm}(\psi_0 t + C_1)\right]^{\frac{1}{\mathcal{B}^{\pm}}}, \tag{21}$$

where C_1 is an integral constant and the expression $\mathcal{B}^{\pm}$ is introduced for simplicity as

$$\mathcal{B}^{\pm} = \frac{5\left(\omega + \frac{6}{5}\alpha\right)\mathcal{A}^{\pm} - 6(\omega + 2\alpha)\left(\omega + \frac{3}{2}\alpha\right)}{\omega(\mathcal{A}^{\pm} - 3\alpha - 2\omega)}. \tag{22}$$

Subsequently, using Eq. (21) in Eq. (18) we find

$$\psi^{\pm}(t) = \psi_0\left[\mathcal{B}^{\pm}(\psi_0 t + C_1)\right]^{\frac{-\mathcal{A}^{\pm}+3\alpha}{\omega\mathcal{B}^{\pm}}}. \tag{23}$$

As a result, we have derived the exact expression for the scalar field as a function of the coupling constant ω and the deformation function α, along with certain integration constants.

In the subsequent subsection, we extend the analysis to encompass various types of cosmic evolution, characterized by the BD parameter ω, including scenarios such as radiation-dominated or dark energy-like evolution.

3.2. *Analyzing the equation of state function*

In this section, we investigate the equation of state (EoS) function within the context of the proposed q-deformed scalar–tensor framework. This analysis is crucial for understanding the behavior of cosmological fluids under the modified gravitational dynamics introduced by q-deformation. In the absence of ordinary matter, the scalar field ψ effectively behaves as a kinetic fluid component. Importantly, in the vacuum configuration considered here, the scalar sector satisfies an independent conservation equation derived from the modified field equations, namely

$$\dot{\rho}_{\psi} + 3H\left(1 + W_{\psi}^{\pm}\right)\rho_{\psi} = 0, \tag{24}$$

where the EoS parameter is defined as $W_{\psi}^{\pm} = \frac{P_{\psi}}{\rho_{\psi}}$. Since the modified Friedmann equations can be consistently written in an effective GR-like form with a self-conserved scalar fluid, the standard relation

$$W_{\psi}^{\pm} = -1 - \frac{2\dot{H}}{3H^2}, \tag{25}$$

remains valid within the present framework. By applying this expression together with Eq. (21), we obtain

$$W_{\psi}^{\pm} = -1 + \frac{2}{3}\mathcal{B}^{\pm}. \tag{26}$$

We observe that, in this specific case, the EoS parameter is time-independent and depends solely on the deformation function α and the BD parameter ω. This result

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indicates that cosmic evolution is governed predominantly by the intrinsic parameters of the modified gravitational theory rather than by dynamical matter–energy interactions. Such behavior can lead to scenarios where acceleration or deceleration phases are determined inherently by the gravitational model rather than by evolving matter components.

Figure 2 illustrates the behavior of the EoS parameter $W_\psi^\pm$ given in Eq. (26) as a function of the BD coupling parameter ω for several representative values of the deformation function α, where $\alpha = 1$ corresponds to the undeformed (standard BD) limit. This comparison allows us to explicitly identify the deviations induced by q-deformation relative to the conventional BD cosmology (see, e.g. [49]). The analysis reveals that the following conditions are satisfied:

(1) In the limit of $\omega \to 0$; W_ψ^- becomes an undefined function while W_ψ^+ stabilizes at $\frac{1}{3}$,
(2) For the positive values of ω; $1 \leq W_\psi^- \leq \infty$ and $\frac{1}{3} < W_\psi^+ \leq 1$,
(3) For the negative values of ω; $-\infty \leq W_\psi^- \leq -\frac{1}{3}$ and $-\frac{1}{3} \leq W_\psi^+ < \frac{1}{3}$.

Physically, the plot indicates that the q-deformation (encoded in α) acts as a control parameter that shifts the scalar sector between standard cosmological behaviors: $W_\psi \simeq 1/3$ (radiation-like), $W_\psi \simeq 0$ (matter-like), and $W_\psi \simeq -1$ (dark-energy-like). In particular, the W_ψ^- branch accesses the negative-pressure domain ($W_\psi < -1/3$) required for accelerated expansion, whereas the W_ψ^+ branch remains non-accelerating for the same parameter ranges.

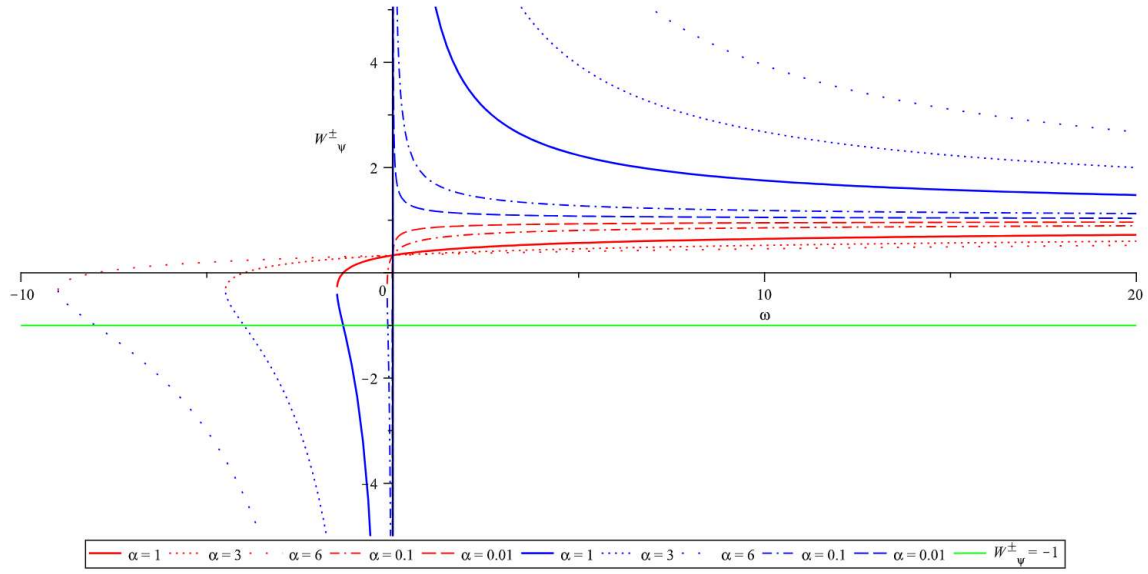

Fig. 2. (Color online) The equations of state functions $W_\psi^\pm$ for the scalar field ψ versus the parameter ω for various values of the deformation function α. The red and blue lines represent W_ψ^+ and W_ψ^- functions, respectively. The green line corresponds to the dark energy-dominated era ($W_\psi^\pm = -1$).

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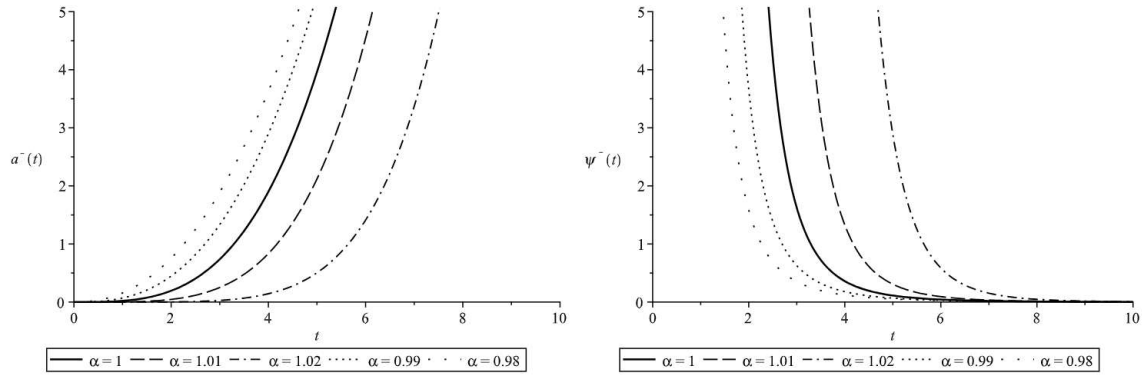

Fig. 3. The time evolution of the scale factor $a(t)^{-}$ and the scalar field $\psi(t)^{-}$ versus time for various values of the deformation function α. Here the constants are set to be $\psi_0 = 1$, $C_1 = 0$, $\omega = -1.4$ and $\alpha = 1$ (solid line) which corresponds to the non-deformed case.

Accordingly, the plots in Fig. 3 exemplify the accelerated dynamics of the universe and illustrate the impact of the deformation function α on both the scale factor $a(t)^{-}$ and the scalar field $\psi(t)^{-}$ within the framework of q-deformed scalar–tensor gravity under specific conditions.

3.3. *Phenomenology and viability in the matter-free sector*

The matter-free solution (21) implies a power-law expansion $a(t)^{\pm} \propto (\psi_0 t + C_1)^{1/\mathcal{B}^{\pm}}$, which yields the Hubble parameter

$$H(t)^{\pm} = \frac{\dot{a}}{a} = \frac{1}{\mathcal{B}^{\pm}(t + C_1/\psi_0)}, \tag{27}$$

then the deceleration parameter becomes

$$q_{\psi}^{\pm} = -\frac{\ddot{a}a}{\dot{a}^2} = \mathcal{B}^{\pm} - 1. \tag{28}$$

where $q_{\psi}^{\pm}$ denotes the deceleration parameter and should not be confused with the deformation parameter q. This equation shows that $q_{\psi}^{\pm}$ is constant and independent of cosmic time. Typically, in cosmological models, deceleration parameter evolves over time, reflecting different expansion phases, such as an early decelerating phase followed by late-time acceleration. However, the solution without matter sources, $q_{\psi}^{\pm}$ is solely determined by the deformation function α and the BD parameter ω, implying that the expansion behavior is fixed once these parameters are set.

Therefore, accelerated expansion requires $\mathcal{B} < 1$ (equivalently $W_{\psi} < -1/3$ via Eq. (26)), which occurs only for the negative-sign branch in Eq. (18) for appropriate values of (ω, α). Similar to the pure BD gravity, this solution requires also $\omega < 0$ for the dark energy era. On the other hand, the Solar System and cosmological observations constrain the BD parameter to be large and positive (typically $\omega \gg 1$) [50–53]. In conventional BD theory, such large values generally suppress scalar-field-driven acceleration in the absence of a potential or additional matter sources.

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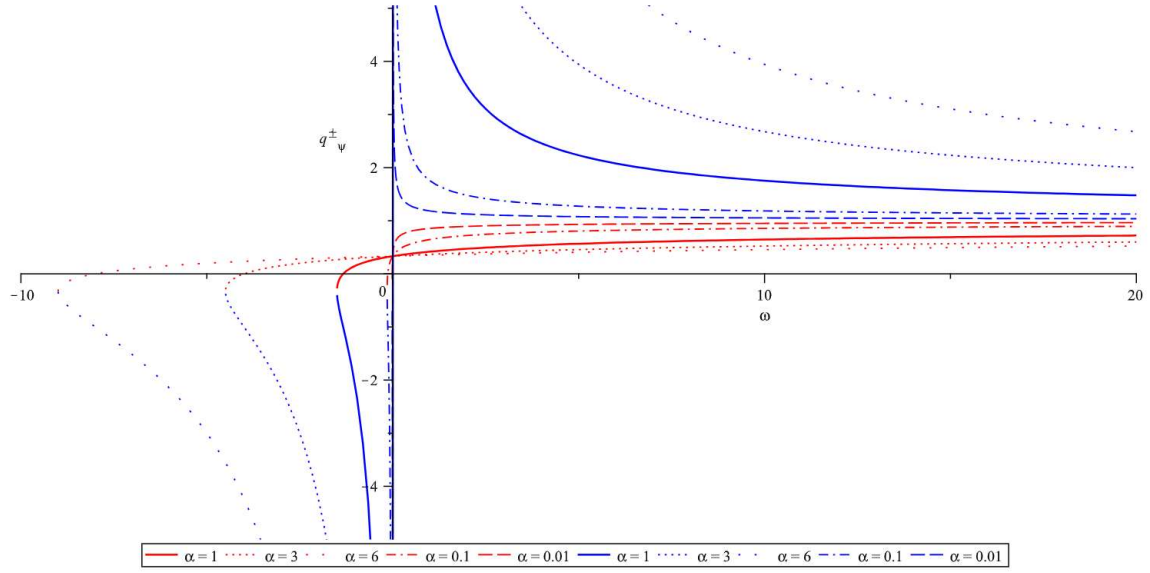

Fig. 4. (Color online) The deceleration parameter $q^{\pm}_{\psi}$ versus the parameter ω for various values of the deformation function α. The red and blue lines represent q^{+}_{ψ} and q^{-}_{ψ} functions, respectively, where $\alpha = 1$ (solid line) which corresponds to the non-deformed case, $\alpha = 3$ (dotted line), $\alpha = 6$ (spaced dotted line), $\alpha = 0.1$ (dash-dot line) and $\alpha = 0.01$ (dashed line).

Therefore a full assessment of viable cosmological solution may include matter, possible scalar self-interactions, additional scalar potential $V(\psi)$ and field-dependent $\omega \to \omega(\psi)$ [54, 55].

A basic viability requirement is the positivity of the effective gravitational coupling in Eq. (6), i.e.

$$\alpha(z, q)\ \psi > 0, \tag{29}$$

together with the reality condition (20). Moreover, the exact solution (23) directly determines the time variation of G_{eff} through $\dot{G}_{\text{eff}}/G_{\text{eff}} = -\dot{\psi}/\psi$ when α is effectively constant. This provides a simple phenomenological handle on the model and motivates a future, more detailed confrontation with local and cosmological constraints.

Figure 4 shows the deceleration parameter $q^{\pm}_{\psi}$ as a function of the BD coupling ω for several fixed values of the deformation function α and therefore provides a compact parameter-space diagnostic for the vacuum solutions: the two branches behave qualitatively differently. The (+) branch typically remains decelerating ($q^{+}_{\psi} > 0$) and tends to an asymptotic value as ω increases, while the (−) branch can enter an accelerating regime ($q^{-}_{\psi} < 0$) for an allowed range of (ω, α).

4. Conclusion

We have investigated a q-deformed generalization of BD gravity, motivated by entropic-gravity considerations, in a spatially flat FLRW background. We derived the modified Friedmann equations, Eqs. (12) and (13), and obtained exact vacuum

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solutions for the scale factor, Eq. (21), and scalar field, Eq. (23). These solutions reduce to the standard BD expressions in the limit $\alpha(z,q) \to 1$ (see Fig. 1).

Interpreting the scalar sector as an effective fluid, we found that the EoS parameter $W_{\psi}^{\pm}$ is time-independent and is controlled only by the deformation function $\alpha(z,q)$ and the BD coupling ω. Depending on the branch and parameter choices, the scalar sector can mimic radiation-, matter-, quintessence-, or phantom-like behavior (see Fig. 2).

For the vacuum solutions studied here, accelerated expansion occurs only for the minus branch (see Eq. (18)) in a restricted region of parameter space which requires $\omega < 0$ (see Figs. 2 and 3) in accordance with the pure BD gravity. We also found that the deceleration parameter $q_{\psi}^{\pm}$ is constant, Eq. (28) (Fig. 4).

However, standard BD cosmology without additional ingredients (e.g. a scalar potential or extra matter couplings) is known to have difficulty generating late-time acceleration for observationally allowed large positive values of ω. Therefore, the matter-free solutions presented here should be viewed primarily as an analytic toy model that isolates the role of $\alpha(z,q)$. Constructing a fully realistic cosmology consistent with current bounds ($\omega \gg 1$) likely requires additional sectors, which we leave for future work.

The present framework also exhibits a close correspondence with effective modified-gravity theories. In particular, since $f(R)$ gravity can be reformulated as a scalar–tensor theory equivalent to a BD-type model, the obtained q-deformed cosmological equations may likewise be interpreted within an effective gravitational theory [56, 57], in which the generalized cosmological dynamics arise from modified entropy functionals. However, unlike purely geometrical $f(R)$-type reconstructions, the present approach retains the BD scalar field as an explicit dynamical degree of freedom through the deformation function $\alpha(z,q)$. Furthermore, recent DESI data strongly favor modified gravity or dynamical dark energy over ΛCDM, indicating an evolving equation of state with a phantom-to-quintessence transition. Specific $f(R)$ models, such as exponential and logarithmic variants, are highly consistent with these observations and statistically outperform the standard model. These scenarios naturally mimic the required dynamical behavior, supporting our framework's interpretation of modified gravity as an effective description of late-time acceleration [58–61].

Overall, the q-deformation modifies the effective gravitational coupling through $\alpha(z,q)$ and leads to qualitatively different scalar-driven expansion histories relative to standard BD cosmology. However, the model does not by itself reproduce the observed transition from an early decelerating era to late-time acceleration. Additionally, a detailed assessment of observational viability, including local-gravity constraints and potential implications for cosmological tensions, requires extending the present vacuum analysis and is left for future work.

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ORCID

Salih Kibaroğlu https://orcid.org/0000-0002-8691-4959
Mustafa Senay https://orcid.org/0000-0002-9480-9040

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