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Total Reduction of Chiral Oscillator and Its Dirac Analysis

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Abstract. Hamiltonian formulation of the dynamical systems generated by the second order degenerate Chiral Lagrangian is studied by reducing into first order Lagrangian and Dirac- Bergmann algorithm has been performed.
Keywords: Second order degenerate Lagrangians, the Dirac-Bergmann algorithm, Chiral Oscillator.
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INTRODUCTION

There exist two different but equivalent representations of the Newton's classical dynamics, namely the Lagrangian and the Hamiltonian dynamics. The passage between the Lagrangian and the Hamiltonian dynamics is available by means of the Legendre transformations if some non-degeneracy conditions holds. If the configuration space is an n-dimensional manifold Q, then the Lagrangian dynamics is generated by a second order Lagrangian function(al) $L(x^i, \dot{x}^i, \ddot{x}^i)$ on the tangent bundle T^2Q [1, 2]. On the other hand, Hamiltonian representation of a physical system is defined on the cotangent bundle T^*TQ of the configuration manifold Q. Transformation from Lagrangian formalism to Hamiltonian formalism can be achieved by the Legendre transformation defined in terms of the fiber derivative. A higher dimensional version the fiber derivative was introduced by Ostrogradsky around 1850's [19]. If the Lagrangian is non-degenerate in the sense of Ostrogradski, then the Legendre transformation is immediate. If it is degenerate, then one additionally needs to employ the Dirac-Bergmann algorithm [8, 9, 12, 13].

In our previous work [6], Hamiltonian analysis of second order Lagrangians were presented by recasting the second order Lagrangians into first order forms on some iterated tangent bundles. Such reduction procedures are far from being unique and are mainly achieved by enlarging the configuration space with introductions of new coordinates and Lagrange multipliers. In reduction procedures, the reduced first order Lagrangian would be degenerate because of presence of the Lagrange multipliers even if the second order one is not.

As an application of second order degenerate theory, we will study 2-dimensional non-relativistic Chiral oscillator with a Chern-Simons like term [3, 4], whose dynamics is described by the degenerate Lagrangian

$$L(x^i, \dot{x}^i, \ddot{x}^i) = -\frac{\lambda}{2}\epsilon_{ij}\dot{x}^i\ddot{x}^j + \frac{m}{2}\dot{x}^i\dot{x}^i \tag{1}$$

where λ and m are non-vanishing constants, $x^i = (x, y)$ and ϵ_{ij} is the Levi-Civita symbol such that $\epsilon_{12} = 1 (i, j = 1, 2)$.

The goal of this paper is to present one of the reduction given in [6] for the Chiral osscillator. Chiral Lagrangian is reduced into a first order Lagrangian by defining a Lagrange multipliers and then for this reduced Lagrangian Dirac-Bergmann algorithm is employed to derive Hamilton equations.

FIRST ORDER LAGRANGIAN THEORY AND DIRAC-BERGMANN ALGORITHM

First order Lagrangian $L(x^i, \dot{x}^i)$ is a real valued function on the tangent bundle TQ, and generates the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0. \tag{2}$$

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In order to go over from the Lagrangian to the Hamiltonian formulation, one should introduce generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{x}^i}, \quad i = 1, ...n \tag{3}$$

conjugated to the generalized coordinates x^i. If the rank r of the hessian matrix $\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$ is less then dimension n of Q, then all of the velocities cannot be solved in terms of the momentum variables. So that, the momentum variables are not all independent. Instead, there exists some relations

$$\Phi_m(x^i, p_i) \approx 0, \quad m = 1, ..., n - r, \tag{4}$$

called as primary constraints, [12, 13] and they constitute a primary constraint submanifold . The equality in the definition of primary constraints is weak in the sense that it is ignored during set up of Dirac formalism.

Hamiltonian $H(x^i, p_i)$ for the degenerate cases is far from being unique. One may add arbitrary linear combinations of the primary constraints to the Hamiltonian function. This leads to the total Hamiltonian

$$H_T(x^i, p_i) = H(x^i, p_i) + u^m(x^i, p_i)\Phi_m(x^i, p_i) \tag{5}$$

for arbitrary functions u^m called the Lagrange multipliers [12, 13].

The requirement that solutions of Euler-Lagrange equations remain on constraint submanifold is described by the weak equality

$$\dot{\Phi}_\beta = \{\Phi_\beta, H\} + u^\alpha\{\Phi_\beta, \Phi_\alpha\} \approx 0, \quad \beta = 1, ..., n - r, \tag{6}$$

that is, equality modulo primary constraints. These so called consistency conditions may lead to determination of Lagrange multipliers if the left hand sides contain u^α. In this case, one solves for u^α through the set of linear equations

$$\{\Phi_\beta, \Phi_\alpha\}u^\alpha = -\{\Phi_\beta, H\}$$

for which, the solution set, namely, number of multipliers that can be solved is characterized by the rank of the skew-symmetric matrix $\{\Phi_\beta, \Phi_\alpha\}$ of Poisson brackets. Obviously, if the number of primary constraints is odd u^αs cannot be solved completely and one expects more constraints to determine H_T in terms of phase space variables. This secondary constraints follow if left hand sides do not contain u^α or, $n - r$ is odd. Repeating this process, one enlarges the primary constraint set with the new (secondary, tertiary, etc.) constraints, redefines H_T by introducing new Lagrange multipliers for new constraints and, repeats the consistency computations.

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SECOND ORDER LAGRANGIAN THEORY AND TOTAL REDUCTION

Second order Lagrangian $L(x^i, \dot{x}^i, \ddot{x}^i)$ is a real valued function on the second tangent bundle T^2Q, and generates the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^i} = 0. \tag{7}$$

Given a second order Lagrangian function, define the $3n$-dimensional configuration space $M = T^2Q$ with coordinates

$$q_1^i = x^i, \quad q_2^i = \dot{x}^i, \quad q_3^i = \ddot{x},^i \tag{8}$$

[20, 21]. With this coordinate tranformations, we define the first order Lagrangian function

$$L_C = L\left(q_1^i, q_2^i, q_3^i\right) + \lambda_i\left(\dot{q}_1^i - q_2^i\right) + \beta_i\left(\dot{q}_2^i - q_3^i\right), \tag{9}$$

where $\lambda_i, \beta_i \in \mathbb{R}^n$ are Lagrange multipliers and $\dot{q}_1^i - q_2^i = 0, \dot{q}_2^i - q_3^i = 0$ are the set of constraints. For the first order Lagrangian function L_C in (9), the first order Euler-Lagrange equations (2) obtained by variations of $q_1^i, q_2^i, q_3^i, \lambda_i$ and β_i are

$$\frac{\partial L}{\partial q_1^i} - \dot{\lambda}_i = 0, \qquad \frac{\partial L}{\partial q_2^i} - \lambda_i - \dot{\beta}_i = 0, \qquad \frac{\partial L}{\partial q_3^i} - \beta_i = 0, \qquad \dot{q}_1^i - q_2^i = 0, \qquad \dot{q}_2^i - q_3^i = 0.$$

For the Hamilton formalism, from the definitions of conjugate momenta for the first order Lagrangian (9), the primary constraints are defined as

$$\Phi_i^1 = p_i^1 - \lambda_i, \quad \Phi_i^2 = p_i^2 - \beta_i, \quad \Phi_i^3 = p_i^3, \quad \Phi_i^\lambda = p_i^\lambda, \quad \Phi_i^\beta = p_i^\beta.$$

In this case, the total Hamiltonian function becomes

$$H_T = H + \Phi_i^1 u_1^i + \Phi_i^2 u_2^i + \Phi_i^3 u_3^i + \Phi_i^{(\lambda)} u_\lambda^i + \Phi_i^\beta u_\beta^i \tag{10}$$

where the canonical Hamiltonian is

$$H = -L\left(q_1^i, q_2^i, q_3^i\right) + \lambda_i q_2^i + \beta_i q_3^i.$$

CHIRAL OSCILLATOR

Euler-Lagrange equations (7) for the Chiral Lagrange function in (1) are

$$\lambda \epsilon_{ij} \dddot{x}^j - m\ddot{x}_i = 0 \tag{11}$$

third order instead of fourth order since Chiral Lagrange function in (1) is degenerate.

For the Chiral Lagrange function, the total reduced first order Lagrange function in (9) is

$$L_T(q_1^i, q_2^i, q_3^i) = -\frac{\lambda}{2}\epsilon_{ij}\dot{q}_2{}^i q_3^j + \frac{m}{2}\delta_{ij} q_2^i q_2^j + \alpha_i(\dot{q}_1^i - q_2^i) + \beta_i(\dot{q}_2^i - q_3^i) \tag{12}$$

where α_i and β_i are Lagrange multipliers.

To pass the Hamiltonian formalism , conjugate momenta are defined by

$$p_i^1 = \frac{\partial L_T}{\partial \dot{q}_1^i} = \alpha_i, \quad p_i^2 = \frac{\partial L_T}{\partial \dot{q}_2^i} = \beta_i, \quad p_i^3 = \frac{\partial L_T}{\partial \dot{q}_3^i} = 0, \quad p_\alpha^i = \tfrac{\partial L_T}{\partial \dot{\alpha}^i} = 0, \quad p_\beta^i = \tfrac{\partial L_T}{\partial \dot{\beta}^i} = 0 \tag{13}$$

and Hamiltonian function for the first order Lagrangian function in (12) is

$$H = p_i^1 \dot{q}_1^i + p_i^2 \dot{q}_2^i + p_i^3 \dot{q}_3^i - L_T = \frac{\lambda}{2}\epsilon_{ij} q_2^i q_3^j - \frac{m}{2}\delta_{ij} q_2^i q_2^j + \alpha_i q_2^i + \beta_i q_3^i. \tag{14}$$

Definition of momenta in (13) leads to the set of primary constraints

$$\Phi_i^1 = p_i^1 - \alpha_i, \quad \Phi_i^2 = p_i^2 - \beta_i, \quad \Phi_i^3 = p_i^3, \quad \Phi_\alpha^i = p_\alpha^i, \quad \Phi_\beta^i = p_\beta^i \tag{15}$$

since neither of the momenta are solved for the velocities. Total Hamiltonian is defined as

$$H_T = H + u_1^i \Phi_i^1 + u_2^i \Phi_i^2 + u_3^i \Phi_i^3 + u_i^\alpha \Phi_\alpha^i + u_i^\beta \Phi_\beta^i \tag{16}$$

canonical Hamiltonian function in (14) and primary constraints in (15) multiplied with unknown coefficients $u_1^i, u_2^i, u_3^i, u_i^\alpha$ and u_i^β.

Consistency condition of primary constraints

$$\dot{\Phi}_i^1 = \{\Phi_i^1, H_T\} = -u_i^\alpha, \quad \dot{\Phi}_i^2 = \{\Phi_i^2, H_T\} = -\frac{\lambda}{2}\epsilon_{ij} q_3^j + m\delta_{ij} q_2^j - \alpha_i - u_i^\beta \tag{17}$$

$$\dot{\Phi}_i^3 = \{\Phi_i^3, H_T\} = -\frac{\lambda}{2}\epsilon_{ji} q_2^j - \beta_i, \quad \dot{\Phi}_\alpha^i = \{\Phi_\alpha^i, H_T\} = -q_2^i + u_1^i, \quad \dot{\Phi}_\beta^i = \{\Phi_\beta^i, H_T\} = -q_3^i + u_2^i \tag{18}$$

lead to determine $u_1^i \approx q_2^i, u_2^i \approx q_3^i, u_i^\alpha \approx 0$, and $u_i^\beta \approx -\frac{\lambda}{2}\epsilon_{ij} q_3^j + m\delta_{ij} q_2^j - \alpha_i$ and secondary constraint $\Phi_i = \frac{\lambda}{2}\epsilon_{ji} q_2^j + \beta_i$ arises. Conservation of this secondary constraint

$$\dot{\Phi}_i = \{\Phi_i, H_T\} = -\frac{\lambda}{2}\epsilon_{ji} u_2^j + u_i^\beta \tag{19}$$

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satisfied identically after substituting the determined u_2^j and u_i^β.

Substitution of the determined Lagrange multipliers $u_1^j, u_2^j, u_i^\alpha, u_i^\beta$ into total Hamiltonian function (16) gives the following explicit expression for the total Hamiltonian function

$$H_T = \frac{\lambda}{2}\epsilon_{ij}q_2^i q_3^j - \frac{m}{2}\delta_{ij}q_2^i q_2^j + p_i^1 q_2^i + p_i^2 q_3^i + p_i^3 u_3^i + p_\beta^i\left(-\frac{\lambda}{2}\epsilon_{ij}q_3^j + m\delta_{ij}q_2^j - \alpha_i\right). \tag{20}$$

Hamilton equations of motion governed by the total Hamiltonian in (20) are

$$\dot{q}_1^i \approx q_2^i, \quad \dot{q}_2^i \approx q_3^i, \quad \dot{q}_3^i \approx u_3^i, \quad \dot{\alpha}_i \approx 0, \quad \dot{\beta}_i \approx -\frac{\lambda}{2}\epsilon_{ij}q_3^j + m\delta_{ij}q_2^j - \alpha_i \tag{21}$$

$$\dot{p}_i^1 \approx 0, \quad \dot{p}_i^2 \approx -\frac{\lambda}{2}\epsilon_{ij}q_3^j + m\delta_{ij}q_2^j - p_i^1, \quad \dot{p}_i^3 \approx -p_i^2 - \frac{\lambda}{2}\epsilon_{ji}q_2^j, \quad \dot{p}_\alpha^i \approx 0, \quad \dot{p}_\beta^i \approx 0. \tag{22}$$

From these equations $\dot{p}_i^1$ and $\dot{\alpha}_i$ gives the Euler-Lagrange equation (11) , the remaining equations are satisfied identically.

CONCLUSION

In this study, the reduction of Chiral Lagrangian to the first order theory has been presented. This reduced Chiral Lagrangian has two kinds of degeneracies, one is due to the reduction other is due to the functional structure of the oscillator. The fiber derivatives have been computed and the Dirac-Bergmann algorithm has been performed for the reduced Chiral Lagrangian. At the end Hamilton equations of motion have been derived. These equations are equivalent to Euler Lagrange equations. On the other hand, in the former paper [5], reduction of Chiral Lagrangian into another first order forms and their Hamiltonian formalisms were obtained. Hamilton equations that are derived in this article and in [5] are identical.

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